An extension of van Lambalgen's Theorem to infinitely many relative 1-random reals

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Abstract

Van Lambalgen's Theorem plays an important role in algorithmic randomness, especially when studying relative randomness. In this paper we extend van Lambalgen's Theorem by considerint the join of infinitely many reals which are random relative to each other.

In addition, we study computability of the reals in the range of Omega operators. It is known that $\Omega^{\phi'}$ is high. We extend this result to that $\Omega^{\phi^{(n)}}$ is high$_n$. We also prove that there exists A such that, for each n, the real Ω^A_M is high$_n$ for some universal Turing machine M by using the extended van Lambalgen's Theorem.

1 Introduction

Van Lambalgen's Theorem provides a strong connection between randomness and computability and it is a very powerful tool to study computability and randomness. In this paper we extend this theorem to infinitely many relative random reals.

In addition, we study computability of the reals in the range of Omega operators. We use the extended van Lambalgen's Theorem in proving that there exists a real A such that for each n, the real Ω^A_M is high$_n$ for some universal Turing machine M.

In section 3 we prove two properties of martingales. This is because we extend van Lambalgen's Theorem by martingales. One property we prove here is a saving lemma for c.e. martingales and the other is about h-order martingales. It is known that saving lemmas for computable or resource-bounded martingales hold, but the proof can not be adapted to c.e. martingales. Here we prove a saving lemma for c.e. martingales. Again, this proof can not be adapted to computable or resource-bounded martingales.

In section 4 we define partial strings and expand the domain of martingales from strings to the partial strings. By using these extended martingales we can strengthen the saving lemma.

In section 5 we study van Lambalgen's Theorem. Van Lambalgen's Theorem is about two relative random reals and can deal with finitely many relative random reals. We extend this theorem to infinitely many relative random reals.

In section 6 we prove some results about the computability of the reals in the range of Omega operators. It is known that $\Omega^{\phi'}$ is high. We extend this to that $\Omega^{\phi^{(n)}}$ is high$_n$.

We also prove that there exists A such that, for each n, the real $\Omega_M^A = \Omega^{\phi^{(n)}}$ for some universal prefix-free Turing machine M.

2 Preliminaries

Now we look at notations we use in this paper and basic definitions. For a more complete introduction, see Soare [12] or Odifreddi [9, 10] for computability theory and Li and Vitányi [6], Downy and Hirschfeldt [3] or Nies [8] for algorithmic randomness.

We say that ψ is a *partial computable function* from $\mathbb{N}^k$ to $\mathbb{N}$ if there is a Turing machine P such that $\psi(x_0, \cdots, x_{k-1}) = y$ iff P on inputs $x_0, \cdots, x_{k-1}$ outputs y. For a set A of natural numbers, the set $A' = \{e \mid \Phi_e^A(e)\}$ is called the *jump* of A where Φ_e^A is the e-th partial computable function $\mathbb{N} \to \mathbb{N}$ with A as an oracle. We write $A^{(n)}$ to mean n-th jump of A. We say that A is T-reducible to B, written as $A \leq_T B$, if $A = \Phi_e^B$ for some e.

We can regard a set A as an infinite binary sequence such that i-th bit of the sequence is 1 if $i \in A$ and 0 if $i \notin A$. The Cantor space, denoted by 2^ω, is the set of all infinite binary sequences and $2^{<\omega}$ denotes the set of all finite binary strings. We also identify real numbers with their inifinite binary expansion. Elements of Cantor space 2^ω are sometimes called reals. We say that A is B-c.e. real if A is the limit of a B-computable rational approximation. A function f is c.e. if the values $f(n)$ are uniformly c.e. reals.

A *Martin-Löf test* is a sequence of uniformly c.e. open sets $\{U_n\}$ such that $\mu(U_n) \leq 2^{-n}$ where μ is the uniform measure on Cantor space. A real A passes a Martin-Löf test U_n if $A \notin \bigcap_n U_n$. A real A is *Martin-Löf random* or *1-random* if A passes all Martin-Löf tests.

We identify $\sigma \in 2^{<\omega}$ with $n \in \mathbb{N}$ such that the binary representation of $n+1$ is 1σ. A string is an element of $2^{<\omega}$. For any $\sigma \in 2^{<\omega}$, $|\sigma|$ denotes the length of σ. We write $\sigma(i)$ for the $(i+1)$-th bit of σ. Let λ denote the empty string. A set X of strings is *prefix-free* if whenever $\sigma, \tau \in X$, then σ is not a proper prefix of τ. A partial computable function $M : 2^{<\omega} \to 2^{<\omega}$ is called a *prefix-free machine* if $\mathrm{dom}(M)$ is prefix-free. There is a *universal* prefix-free machine, i.e., a prefix-free machine U such that for each prefix-free machine M there is a string $\tau \in 2^{<\omega}$ for which $(\forall\sigma)U(\tau\sigma) = M(\sigma)$ or both $U(\tau\sigma)$ and $M(\tau)$ diverge. Let $\Omega_U^A = \sum_\sigma 2^{-|\sigma|} [\![U^A(\sigma)\downarrow]\!]$. This is called halting probability relative to A. We can regard Ω_U as an operator from reals to reals. When U is universal prefix-free machine, Ω_U is called an Omega operator via U. Clearly Ω_U^A is an A-c.e. real for each prefix-free machine U.

3 Martingales

In this section we recall some definitions related to martingales and prove some results needed in the next section.

3.1 Definitions and basic properties

A *martingale* is a function $d : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ that satisfies the following fairness condition: for every $\sigma \in 2^{<\omega}$

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

The martingale d *succeeds* on a real A if $d(A) = \sup_n d(A \upharpoonright n) = \infty$. The success set $\mathrm{Succ}(d)$ is the set of reals on which d succeeds.

The following is a very important classical result by Schnorr [11].

Theorem 3.1. *A real is 1-random if no c.e. martingale succeeds on it.*

A martingale d is *universal* if $\mathrm{Succ}(d') \subseteq \mathrm{Succ}(d)$ for each c.e. martingale d'. Note that if d is a universal c.e. martingale then a real is 1-random iff d doesn't succeed on it. It is well-known that there is a universal c.e. martingale [11].

For another example of a martingale, which is simple and useful, let us define a conditional probability B_U. For a c.e. open set U, $B_U(\sigma) = 2^{|\sigma|}\mu(U \cap [\sigma])$. Intuitively this is the chance to get into U when starting from σ. In particular, $B_U(\lambda) = \mu(U)$. If $[\sigma] \subseteq U$ then $B_U(\sigma) = 1$. Although this martingale doesn't succeed on any reals, it is useful to assemble complex martingales from simple ones. Generally if d_i is a c.e. martingale for each i and $\sum_i d_i(\lambda) < \infty$ then $d = \sum_i d_i$ is a c.e. martingale [8].

The following assertion is also a very important classical lemma.

Lemma 3.2 (Kolmogorov's inequality, see Ville [14])**.** *Let d be a martingale. Let $S^k_\sigma(d) = \{\tau \mid d(\tau) \geq k$ and $\sigma \preceq \tau\}$, then*

$$2^{-|\sigma|}d(\sigma) \geq \mu(S^k_\sigma(d))k.$$

In particular $\mu(S^k_\lambda(d)) \leq d(\lambda)k^{-1}$.

3.2 Saving lemma

Next we study a saving lemma or slow-but-sure-winning lemma. Similar results have appeared in various forms in the literature [7, 1]. It essentially says that we can assume that a martingale grows almost monotonically (sure winnings) although it is slow.

Saving lemmas for martingales has appeared in various forms [7, 1, 2]. The idea of the proof is that in the betting of the betting strategy, every time your capital increases to more than 2, you take 1 or 2 from your capital and "keep it in the bank" and only continue betting with the remaining little bit of capital. If the original betting strategy succeeds, then infinitely often the little bit of capital you are betting with will increase above so that this betting strategy succeeds as well. The proof can be adapted straightforwardly to computable martingales and resource-bounded ones.

On the contrary, the proof can not be adapted to c.e. martingales. It is because one needs to divide by $d(\sigma)$ to get the saving martingale d', which may not be a c.e. martingale. One can prove a saving lemma for c.e. supermartingales by similar ideas. Here we prove a saving lemma of c.e. martingales by another idea. However this proof can not be adapted to computable martingales or resource-bounded martingales.

Lemma 3.3 (Saving lemma for c.e. martingales)**.** *Let d be a c.e martingale. Then there is a c.e. martingale d' with $Succ(d) \subseteq Succ(d')$ and a constant c such that*

$$(\forall\sigma)(\forall\tau)[d'(\sigma\tau) > d'(\sigma) - c].$$

We say that d' is saving if this condition holds.

Proof. We can assume $d(\lambda) \leq 1$. Let $U_n = \{[\sigma] \mid d(\sigma) \geq 2^n\}$ and d_n be a conditional probability of U_n, i.e., $d_n(\sigma) = 2^{|\sigma|}\mu(U_n \cap [\sigma])$. Let

$$d'(\sigma) = \sum_n d_n(\sigma).$$

Note that $d_n(\lambda) = \mu(U_n) \leq 2^{-n}$. Hence $d'(\lambda) = \sum_n d_n(\lambda) \leq 2$. Since each $d_n(\sigma)$ is c.e. uniformly in n and σ, we get d' is a c.e. martingale.

We shall prove $\mathrm{Succ}(d) \subseteq \mathrm{Succ}(d')$. Let A be a real. Suppose d succeeds on A. For all k there exists m such that $d(A \upharpoonright m) \geq 2^k$. Hence $[A \upharpoonright m] \in U_k$. It follows that $d_l(A \upharpoonright m) = 2^m\mu([A \upharpoonright m]) = 1$ for each $n \leq k$. Hence $d'(A \upharpoonright m) = \sum_n d_n(A \upharpoonright m) \geq k$. Since k is arbitrary, d' also succeeds on A.

Next we shall prove that d' is saving, i.e., $(\forall\sigma)(\forall\tau)[d'(\sigma\tau) > d'(\sigma) - c]$. Fix σ, τ. Let $m = \max\{n \mid [\sigma] \subseteq U_n\}$. Note that $d(\sigma) < 2^{m+1}$ otherwise $[\sigma] \subseteq U_{m+1}$. Let

$$e(\sigma) = \sum_{n=0}^{m} d_n(\sigma) \text{ and } f(\sigma) = \sum_{n=m+1}^{\infty} d_n(\sigma).$$

Note that $d_n(\sigma) = 1$ for all $n \leq m$. Moreover $d_n(\sigma\tau) = 1$ for all $n \leq m$ and τ. Hence $e(\sigma) = m = e(\sigma\tau)$. On the contrary $2^{-|\sigma|}d(\sigma) \geq \mu(U_n \cap [\sigma])2^n$ by Kolmogorov inequality. Hence

$$d_n(\sigma) = 2^{|\sigma|}\mu(U_n \cap [\sigma]) \leq d(\sigma)2^{-n} \leq 2^{m+1-n}.$$

So

$$f(\sigma) = \sum_{n=m+1}^{\infty} d_n(\sigma) \leq \sum_{n=m+1}^{\infty} 2^{m+1-n} \leq 2.$$

Now

$$d'(\sigma\tau) = e(\sigma\tau) + f(\sigma\tau) \geq e(\sigma\tau) \geq e(\sigma) = d'(\sigma) - f(\sigma) \geq d'(\sigma) - 2.$$

□

3.3 *h*-order martingales

In this subsection we shall prove that a martingale satisfies a fairness condition in any order.

Definition 3.4. *Let d be a c.e. martingale. For an injective function h and $\sigma \in 2^{<\omega}$, let $n_\sigma = \max\{h(i) \mid i < |\sigma|\}$. We define h-order martingale of d as*

$$f_{h,d}(\sigma) = \sum_\tau d(\tau)2^{|\sigma|-|\tau|} [\![\, |\tau| = n_\sigma \text{ and } \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \,]\!].$$

Lemma 3.5. *For each martingale d and each function h, h-order martingale $f_{h,d}$ of d is a martingale. If d is c.e. and h is computable, $f_{h,d}$ is a c.e. martingale.*

Proof. For ease of presentation we drop the subscript h, d from $f_{h,d}$ in this proof.

We prove n_σ is not essential in $f(\sigma)$: i.e., $f(\sigma)$ has the same value by replacing n_σ with a larger number.

Claim1. $f(\sigma) = \sum_\tau d(\tau)2^{|\sigma|-|\tau|}$ $[\![$ $|\tau| = m$ and $\tau(h(i)) = \sigma(i)$ for all $i < |\sigma|$ $]\!]$ for all $m \geq n_\sigma$.

Suppose $m \geq n_\sigma = \max\{h(i) \mid i < |\sigma|\}$. Then

$$\begin{aligned}
&\sum_\tau d(\tau)2^{|\sigma|-|\tau|} [\![\ |\tau| = m+1 \text{ and } \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \ ]\!] \\
=&\sum_\tau d(\tau 0)2^{|\sigma|-|\tau 0|} [\![\ |\tau| = m \text{ and } \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \ ]\!] \\
&+ \sum_\tau d(\tau 1)2^{|\sigma|-|\tau 1|} [\![\ |\tau| = m \text{ and } \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \ ]\!] \\
=&\sum_\tau 2d(\tau)2^{|\sigma|-|\tau|-1} \\
=&\sum_\tau d(\tau)2^{|\sigma|-|\tau|}.
\end{aligned}$$

Claim2. f is a martingale.

Let m be large enough.

$$\begin{aligned}
2f(\sigma) =& \sum_m d(\tau)2^{|\sigma|+1-|\tau|} [\![\ \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \ ]\!] \\
=& \sum_m d(\tau)2^{|\sigma 0|-|\tau|} [\![\ \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \text{ and } \tau(h(|\sigma|)) = 0 \ ]\!] \\
&+ \sum_m d(\tau)2^{|\sigma 1|-|\tau|} [\![\ \tau(h(i)) = \sigma(i) \text{ for all } i < |\sigma| \text{ and } \tau(h(|\sigma|)) = 1 \ ]\!] \\
=& f(\sigma 0) + f(\sigma 1).
\end{aligned}$$

If h is computable, n_σ is computable from h and σ. Hence $f_{h,d}$ is a c.e. martingale. □

4 Martingales on partial strings

We say that x is a partial string if x is a partial function $\mathbb{N} \to \{0, 1\}$ with a finite domain. We define length of x as $|x| = \#\{k \mid x(k) \downarrow\}$ and total length as $\|x\| = \max\{k \mid x(k) \downarrow\} + 1$. For a partial string x and $y \in 2^{<\omega} \cup 2^\omega$, we write $x \sqsubseteq y$ if $x(i) \downarrow \Rightarrow y(i) \downarrow = x(i)$ for all i. Let $[x]$ denote $\{y \in 2^\omega \mid x \sqsubseteq y\}$. Note that $\mu([x]) = 2^{-|x|}$. In the following, x, y, z, w denote partial strings and σ, τ, η denote strings.

Definition 4.1. *Let d be a martingale. We define a martingale on partial strings of d as follows:*

$$\hat{d}(x) = \sum_\sigma d(\sigma)2^{|x|-|\sigma|} [\![\ |\sigma| = \|x\| \text{ and } x \sqsubseteq \sigma \ ]\!].$$

If σ is a string and not partial, $\hat{d}(\sigma) = d(\sigma)$. Then we identify $\hat{d}$ and d.

Note that the length of σ is not essential by the same proof in Lemma 3.5.

Proposition 4.2. *For each $m \geq \|x\|$, $\hat{d}(x) = \sum_\sigma d(\sigma) 2^{|x|-|\sigma|} [\![\, |\sigma| = m$ and $x \sqsubseteq \sigma \,]\!]$.*

Similar to a usual martingale, we can prove Kolmogorov inequality and saving lemma for martingales on partial strings.

Lemma 4.3 (Kolmogorov inequality for martingales on partial strings.)**.** *Let x be a partial string and d a maritingale. For $k \in \mathbb{N}$, let $S^k_x(d) = \{y \mid d(y) \geq k$ and $x \sqsubseteq y\}$. Then $2^{-|x|} d(x) \geq \mu(S^k_x(d))k$.*

Proof. Let X be a prefix-free set of partial strings such that

$$X = \{y \mid d(y) \geq k,\ x \sqsubseteq y \text{ and } d(z) < k \text{ for all } z \sqsubset y\}.$$

Note that X may not be a c.e. set. Clearly $\mu(S^k_x(d)) = \sum_{y \in X} 2^{-|y|} := \mu(X)$. For each $y \in X$,

$$2^{-|y|} k \leq 2^{-|y|} d(y) = \sum_{\sigma \in Y_y} d(\sigma) 2^{-|\sigma|} \tag{1}$$

where $Y_y = \{\sigma \mid |\sigma| = \max\{\|x\|, \|y\|\} \text{ and } y \sqsubseteq \sigma\}$. On the contrary

$$2^{-|x|} d(x) = \sum_{\tau \in Z} d(\tau) 2^{-|\tau|} \tag{2}$$

where $Z = \{\tau \mid |\tau| = \|x\| \text{ and } x \sqsubseteq \tau\}$.

For each $y \in X$ and $\sigma \in Y_y$, there exists $\tau \in Z$ such that $\tau \sqsubseteq \sigma$. Let $W_\tau = \{\sigma \mid \sigma \in Y_y \text{ for some } y \in X \text{ and } \tau \sqsubseteq \sigma\}$. Then $\cup_{y \in X} Y_y = \cup_{\tau \in Z} W_\tau$.

By Kolmogorov inequality, for each $\tau \in Z$,

$$2^{-|\tau|} d(\tau) \geq \sum_{\sigma \in Y_\tau} 2^{-|\sigma|} d(\sigma). \tag{3}$$

Hence

$$\begin{aligned}
2^{-|x|} d(x) &= \sum_{\tau \in Z} d(\tau) 2^{-|\tau|} && \text{by (2)} \\
&\geq \sum_{\tau \in Z} \sum_{\sigma \in W_\tau} 2^{-|\sigma|} d(\sigma) && \text{by (3)} \\
&= \sum_{y \in X} \sum_{\sigma \in Y_y} 2^{-|\sigma|} d(\sigma) \\
&\geq \sum_{y \in X} 2^{-|y|} k && \text{by (1)} \\
&= \mu(S^k_x(d)) k
\end{aligned}$$

□

Lemma 4.4 (Strong saving lemma)**.** *Let d be a c.e martingale. Then there is a c.e. partial martingale d' with Succ(d) $\subseteq$ Succ(d') and a constant c such that*

$$(\forall x)(\forall y)[x \sqsubseteq y \Rightarrow d'(y) > d'(x) - c].$$

We say d' is strong saving if this condition holds.

Proof. The proof is similar to the saving lemma. We can assume $d(\lambda) \leq 1$. Let $U_n = \{x \mid d(x) \geq 2^n\}$. Note that x is a partial string.

Let d_n be a conditional probability of U_n: i.e.,

$$d_n(\sigma) = 2^{|\sigma|}\mu(U_n \cap [\sigma])$$

and

$$d'(\sigma) = \sum_n d_n(\sigma).$$

By Kolmogorov inequality for partial martingales $d_n(\lambda) = \mu(U_n) \leq 2^{-n}$. Hence $d'(\lambda) = \sum_n d_n(\lambda) \leq 2$. Since each $d_n(\sigma)$ is c.e. uniformly in n and σ, we get d' is a uniformly c.e. martingale.

Next we shall prove that d' is strong saving, i.e.,

$$(\forall x)(\forall y)[x \sqsubseteq y \Rightarrow d'(y) > d'(x) - c].$$

Fix x, y for partial strings. Let $m = \max\{n \mid [x] \subseteq U_n\}$. Note that $d(x) < 2^{m+1}$ as otherwise $[x] \subseteq U_{m+1}$. Let

$$e(x) = \sum_{n=0}^{m} d_n(x) \text{ and } f(x) = \sum_{n=m+1}^{\infty} d_n(x).$$

Note that $d_n(x) = 1$ for all $n \leq m$. Moreover $d_n(y) = 1$ for all $n \leq m$ and $y \sqsupseteq x$. Hence $e(x) = m = e(y)$. On the contrary $2^{-|x|}d(x) \geq \mu(U_n \cap [x])2^n$ by Kolmogorov inequality for partial martingales. Hence

$$d_n(x) = 2^{|x|}\mu(U_n \cap [x]) \leq d(x)2^{-n} \leq 2^{m+1-n}.$$

So

$$f(x) = \sum_{n=m+1}^{\infty} d_n(x) \leq \sum_{n=m+1}^{\infty} 2^{m+1-n} \leq 2.$$

Now

$$d'(y) = e(y) + f(y) \geq e(y) \geq e(x) = d'(x) - f(x) \geq d'(x) - 2$$

The proof of Succ(d) $\subseteq$ Succ(d') is the exactly same as that of Lemma 3.3. □

For an injective function h and a string σ, we define σ_h as $\sigma_h(h(i)) = \sigma(i)$ for all i.

Corollary 4.5. *Let $f_{h,d}(\sigma) = d(\sigma_h)$ and we call $f_{h,d}$ the h-order martingale of d. If d is strong saving, h-order martingale of d is also strong saving.*

5 An extension of van Lambalgen's Theorem

In this section we shall extend van Lambalgen's theorem to infinitely many relative 1-random reals. Intuitively Van Lambalgen's Theorem says that any part of the 1-random real does not have information about the other part. Recall that $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$.

Theorem 5.1 (van Lambalgen's Theorem [13])**.** *For every $A, B \in 2^\omega$,*

$$A \oplus B \text{ is 1-random} \iff A \text{ is 1-random and } B \text{ is } A\text{-random.}$$

We refer the reader to [3] for a proof.

We prepare some notations.

Definition 5.2. *Let $\langle m, n\rangle$ be a pair function defined as $m + (m+n)(m+n+1)/2$. Let $\oplus_{i=0}^n A_i = (\cdots((A_0 \oplus A_1)A_2 \cdots) \oplus A_n)$. We define $[\oplus]_{i=0}^\infty A_i = A$ as $A(\langle m, k\rangle) = A_m(k)$ for all k, m. We also define $d(A \oplus \lambda) = \sup_n d(A \upharpoonright n \oplus \lambda)$.*

Theorem 5.3. *There exists a sequence of martingales $\{d_n\}$ such that $[\oplus]_{i=0}^\infty A_i$ is 1-random iff $\sup_n d_n(\oplus_{i=0}^n A_i) < \infty$.*

Proof. Let $A = [\oplus]_{i=0}^\infty A_i$ and $B_n = \oplus_{i=0}^n A_i$. We define n-th pair function $\langle\langle m, k\rangle\rangle_n$ as $B_n(\langle\langle m, k\rangle\rangle_n) = A_m(k)$ for all m, k.

Let d be a c.e. universal strong saving martingale and h_n be uniformly computable functions such that $h_n(\langle\langle m, k\rangle\rangle_n) = \langle m, k\rangle$. We claim that $\{d_{h_n}\}$ satisfies the above condition.

Suppose $d(A) < c$ for a constant c. By Lemma 4.5,

$$\max\{d(x) \mid x \sqsubseteq A\} < d(A) + c'.$$

Since $d_n(B_n \upharpoonright m) = d((B_n \upharpoonright m)_{h_n})$ for all m,

$$\sup_n d_n(B_n) \le \max\{d(x) \mid x \sqsubseteq A\} < c + c'.$$

We shall prove the other direction. Suppose $d([\oplus]_{i=0}^\infty A_i) = \infty$. For all N there exists m such that $d([\oplus]_{i=0}^\infty A_i \upharpoonright m) > N$. Since m is finite, there exists l such that $[\oplus]_{i=0}^\infty A_i \upharpoonright m = [\oplus]_{i=0}^l A_i \upharpoonright m$. Note that d_l is strong saving. Hence $d_l(\oplus_{i=0}^l A_i) > N - c$.

Moreover $(\oplus_{i=0}^n A_i \upharpoonright m)_{h_n} = ((\oplus_{i=0}^n A_i \upharpoonright m) \oplus \lambda)$. □

Remark 5.4. $d_n(\oplus_{i=0}^n A_i) = d_{n+1}((\oplus_{i=0}^n A_i) \oplus \lambda)$ *because* $(B_n \upharpoonright m)_{h_n} = ((B_n \upharpoonright m) \oplus \lambda)_{h_{n+1}}$.

We shall prove that if a martingale does not succeed on a real then you can make sup of the martingale small by replacing initial segment of the real. We write $X =^* Y$ if $X \triangle Y = (X - Y) \cup (Y - X)$ is finite.

Theorem 5.5. *Let d be a c.e. martingale such that $d(\lambda) \le 1$. For a 1-random real A and a computable real $\epsilon > 1$ there exists $B =^* A$ such that $d(B) \le \epsilon$.*

Proof. Let

$$V(\sigma) = \{\sigma\tau \mid d(\hat{\sigma}\tau) \geq \epsilon \text{ for all } \hat{\sigma} \text{ such that } |\hat{\sigma}| = |\sigma|\}.$$

Note that

$$\mu(V(\sigma)) = \mu(V(\hat{\sigma}))$$

for all $\hat{\sigma}$ such that $|\hat{\sigma}| = |\sigma|$. By Kolmogorov inequality

$$2^{-|\sigma|}d(\sigma) \geq \mu(V(\sigma))\epsilon.$$

Hence

$$\mu(V(\sigma))\epsilon \leq \min\{2^{-|\hat{\sigma}|}d(\hat{\sigma}) \mid |\hat{\sigma}| = |\sigma|\} \leq 2^{-|\sigma|}.$$

Let $U_0 = V(\lambda)$ and

$$U_{n+1} = \{\sigma\tau \mid [\sigma] \in U_n \text{ and } [\sigma\tau] \in V(\sigma)\}$$

recursively. Let X be a prefix-free set such that $[X] = [U_n]$. Since $U_{n+1} = \bigcup_{\sigma\in X} V(\sigma)$,

$$\mu(U_{n+1}) = \sum_{\sigma\in X}\mu(V(\sigma)) \leq \sum_{\sigma\in X} 2^{-|\sigma|}\epsilon^{-1} \leq \epsilon^{-1}\mu(U_n).$$

Let $f(n) = \min\{k \mid \epsilon^{-k} \leq 2^{-n}\}$. Then $\{U_{f(n)}\}$ is a Martin-Löf test.

Suppose $d(B) > \epsilon$ for all $B =^* A$ for a contradiction. If $A \in [\sigma]$ then $A \in V(\sigma)$ by definition. It is obvious that $A \in U_0$. Suppose $A \in U_n$. Let σ such that $A \in [\sigma] \subseteq U_n$. Then $A \in V(\sigma)$. Hence $A \in U_{n+1}$. By induction $A \in \bigcap_n U_n$. This is a contradiction with that A is 1-random. □

Theorem 5.6. *Let d be a c.e. martingale and A, B be reals such that $A\oplus B$ is 1-random. For computable reals c such that $d(A \oplus \lambda) \leq c$ and $\epsilon > 0$, there exists $D =^* B$ such that $d(A \oplus D) \leq \epsilon c$.*

Proof. Let

$$X_n(\sigma) = \{\sigma\tau \mid d(A \upharpoonright n \oplus \sigma\tau) \geq \epsilon c\}$$

and

$$W_n(\sigma) = X_{n+1}(\sigma) - X_n(\sigma).$$

By Kolmogorov inequality

$$2^{-n-|\sigma|}d(A \upharpoonright n \oplus \sigma) \geq \mu(W_n(\sigma))\epsilon c.$$

Hence

$$\sum_{n=1}^{\infty} 2^{-n-|\sigma|}d(A \upharpoonright n \oplus \sigma) \geq \sum_{n=1}^{\infty}\mu(W_n(\sigma))\epsilon c.$$

It follows that

$$2^{-|\sigma|}\sup_n\{d(A \upharpoonright n \oplus \sigma)\} \geq \sum_{n=1}^{\infty}\mu(W_n(\sigma)\epsilon c.$$

Let $V(\sigma) = \{\sigma\tau \mid d(A \oplus \hat{\sigma}\tau) > \epsilon c$ for all $\hat{\sigma}$ such that $|\hat{\sigma}| = |\sigma|\}$. Then

$$2^{-|\sigma|} \min_{\sigma} \sup_{n} \{d(A \upharpoonright n \oplus \sigma)\} \geq \sum_{n=1}^{\infty} \mu(W_n(\sigma)\epsilon c.$$
$$2^{-|\sigma|} \sup_{n} d(A \upharpoonright n \oplus \sigma) = \mu(\bigcap_{n} X_n(\sigma))\epsilon c.$$
$$2^{-|\sigma|} = \mu(V(\sigma))\epsilon.$$

The rest of the proof is the same as that of Theorem 5.5

□

Definition 5.7. *A sequence $\{A_n\}$ is relative 1-random if A_n is $\oplus_{i=0}^{n-1} A_i$-random.*

Theorem 5.8. *For a relative 1-random sequence $\{A_n\}$ there exists $\{B_n\}$ such that $B_n =^* A_n$ for each n and $[\oplus]_{i=0}^{\infty} B_i$ is 1-random.*

Proof. Let $\{\epsilon_i\}$ be a sequence of computable reals such that $\Pi_i \epsilon_i < \infty$. Let $\{d_n\}$ be a sequence of c.e. martingales in Theorem 5.3. We only have to construct $\{B_i\}$ such that $d_n(\oplus_{i=0}^{n} B_i) \leq \Pi_{i=0}^{n} \epsilon_i$ by induction.

Since A_0 is 1-random, there exists B_0 such that $d_0(B_0) \leq \epsilon_0$ and $B_0 =^* A_0$ by Theorem 5.5.

Suppose we already constructed B_i for all $i \leq n$. Hence $d_n(\oplus_{i=0}^{n} B_i) \leq \Pi_{i=0}^{n} \epsilon_i$. By remark 5.4 $d_{n+1}((\oplus_{i=0}^{n} B_i) \oplus \lambda) \leq \Pi_{i=0}^{n} \epsilon_i$ Since $(\oplus_{i=0}^{n} B_i) \oplus A_n$ is 1-random, there exists B_n such that $d_{n+1}(\oplus_{i=0}^{n+1} B_i) \leq \Pi_{i=0}^{n+1} \epsilon_i$ by Theorem 5.6. □

6 Computability of the reals in the range of the Omega operator

Next we study computational power of Ω_M^A for a real A.

We recall some results which are needed below.

Theorem 6.1 (Downey, Hirschfeldt, Miller, Nies [4]). *$A' \equiv_T A \oplus \Omega_U^A$, for every $A \in 2^\omega$ and universal prefix-free machine U.*

Theorem 6.2 (Downey, Hirschfeldt, Miller, Nies [4]). *For $A, B \in 2^\omega$, B is an A-c.e. real and A-random iff $B = \Omega_U^A$ for some universal prefix-free oracle machine U.*

Theorem 6.3 (Kučera [5]). *If $A \geq_T \phi'$ then there exists 1-random set B such that $B \equiv_T A$.*

We say that a operator $S : 2^\omega \to 2^\omega$ is *degree invariant* if $A \equiv_T B$ implies $S(A) \equiv_T S(B)$. We know Omega operator is not degree invariant, moreover Ω_M^A may have different T-degree for each M. We write $A \equiv_T B \oplus \Omega^C$ if $B \oplus \Omega_M^C$ is T-equivalent to A for all M.

The order of $\oplus$ does not make the difference in Turing degree, so we abbreviate parenthesis ( ).

We say A is *high* if $\phi'' \leq_T A'$ and A is *high*$_n$ if $\phi^{(n+1)} \leq_T A^{(n)}$.

Recall that A is low for Ω if Ω is A-random. When A is above ϕ', Ω^A is low for Ω, which means that its computational power is weak. However $\Omega^{\phi'}$ is *high*. This is a well-known result.

Theorem 6.4 (see [8])**.** *Let $A = \Omega^{\phi'}$. Then $A' \equiv_T \phi''$, so $\Omega^{\phi'}$ is high.*

This situation can be extended. We prepare a lemma.

Definition 6.5. *Let $R_n = \Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n)}}$.*

Lemma 6.6. *For each n, $R_n \equiv_T \phi^{(n+1)}$ and R_n is 1-random.*

Proof. We use induction.

When $n = 0$ it is obvious because $\phi' \equiv_T \Omega$ and Ω is 1-random.

Suppose R_n is 1-random and $R_n \equiv_T \phi^{(n+1)}$. Then

$$R_{n+1} \equiv_T R_n \oplus \Omega^{\phi^{(n+1)}} \equiv_T \phi^{(n+1)} \oplus \Omega^{\phi^{(n+1)}} \equiv_T \phi^{(n+2)}.$$

Note that $\Omega^{\phi^{(n+1)}}$ is $\phi^{(n+1)}$-random so R_n-random. Since R_n is 1-random, R_{n+1} is 1-random by Theorem 5.1. □

Theorem 6.7. *For each n, $\Omega^{\phi^{(n)}}$ is high$_n$, moreover for $A = \Omega^{\phi^{(n)}}$, the n-th jump $A^{(n)} \equiv_T \phi^{(n+1)}$.*

Proof. First we shall prove that if $m+1 < n$ and $A = R_m \oplus \Omega^{\phi^{(n)}}$, then $A' \equiv_T R_{m+1} \oplus \Omega^{\phi^{(n)}}$.

It is sufficient to show $\Omega^{(m+1)} = \Omega_U^A$ for some U by Theorem 6.1. Since $A \geq_T \phi^{(m+1)}$, $\Omega^{\phi^{(m+1)}}$ is A-c.e. real. Moreover, since R_n is 1-random, $R_{m+1} \oplus \Omega^{\phi^{(n)}}$ is 1-random so R_{m+1} is A-random. Hence $\Omega^{\phi^{(m+1)}} = \Omega_U^A$ for some U by 6.2.

Then we can prove that, for $A = \Omega^{\phi^{(n)}}$, $A^{(n)} \equiv_T \phi^{(n+1)}$. For $B = \Omega_M^{\phi^{(n)}}$, $B^{(n)} \equiv_T \Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n-1)}} \oplus \Omega_M^{\phi^{(n)}} \equiv_T \phi^{(n+1)}$ for each M by the discussion above. □

Lemma 6.8. *If Z is $n+2$-random, there exists a real A and prefix-free universal Turing machine U and V such that $\Omega_U^A = \Omega^{\phi^{(n)}}$ and $\Omega_V^A = Z$.*

Proof. Let $B = (1 + Z - \Omega^{\phi^{(n)}})/2$ and $A = \phi^{(n)} \oplus B$.

First we prove $\Omega^{\phi^{(n)}}$ is A-c.e. real and A-random so that $\Omega_U^A = \Omega^{\phi^{(n)}}$ for some U. Note that $\Omega^{\phi^{(n)}}$ is $\phi^{(n)}$-c.e. real so A-c.e. real. Since Z is R_n-random, B is R_n-random. Hence $\Omega^{\phi^{(n)}}$ is $R_{n-1} \oplus B$-random. Since $R_{n-1} \oplus B \equiv_T A$, $\Omega^{\phi^{(n)}}$ is A-random.

Next we prove Z is A-c.e. real and A-random so that $\Omega_V^A = Z$. Since $\Omega^{\phi^{(n)}}$ is $\phi^{(n)}$-c.e. real, $Z = 2B - 1 + \Omega^{\phi^{(n)}}$ is $A = \phi^{(n)} \oplus B$-c.e. real. On the other hand since Z is R_n-random, R_n is Z-random. Hence $\Omega^{\phi^{(n)}}$ is $\phi^{(n)} \oplus Z$-random. It follows B is $\phi^{(n)} \oplus Z$-random. Hence Z is A-random. □

Theorem 6.9. *There exists a real A such that, for each n, $\Omega_U^A = \Omega^{\phi^{(n)}}$ for some U.*

Proof. We consider 1-random real $R_n = \Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n)}}$. By van Lambalgen Theorem $\Omega^{(n)}$ is $\Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n-1)}}$-random. It follows that

$$A_n = (1 + \Omega^{\phi^{(n)}} - \Omega^{\phi^{(n-1)}})/2 \text{ is } \Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n-1)}}\text{-random.}$$

So once more by van Lambalgen theorem

$$\Omega \oplus \Omega^{\phi'} \oplus \cdots \oplus \Omega^{\phi^{(n-1)}} \oplus A_n \text{ is 1-random.}$$

Similarly let

$$A_k = (1 + \Omega^{\phi^{(k)}} - \Omega^{\phi^{(k-1)}})/2$$

then $\Omega \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is 1-random.

By Theorem 5.8, there exists a 1-random set $C = \Omega \oplus [\oplus]_{i=1}^{\infty} B_i$ and $B_k =^* A_k$.

Let $A = \Omega \oplus [\oplus]_{i=2}^{\infty} B_i$. It is enough to prove that, for each k, $\Omega^{\phi^{(k)}}$ is A-c.e. real and A-random. First $A \geq_T \Omega \geq_T \phi'$ so $\Omega^{\phi'}$ is A-c.e. real. Let $2 \leq m$. Then

$$\sum_{k=2}^{m} A_k = (1 + \Omega^{\phi^{(m)}} - \Omega^{\phi'})/2$$

so

$$\Omega^{\phi^{(m)}} = 2 \sum_{k=2}^{m} A_k + \Omega^{\phi'} - 1$$

which is clearly A-c.e. real.

Since C is 1-random, B_1 is A-random. Note that $B_1 =^* A_1 = 1 + \Omega^{\phi'} - \Omega$ and $A \geq \phi'$. Hence $\Omega^{\phi'}$ is A-random. Moreover

$$\Omega^{\phi^{(k)}} = 2 \sum_{k=2}^{m} A_k + \Omega^{\phi'} - 1$$

is A-random because $\sum_{k=2}^{m} A_k$ can be computed by A. □

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